J. Fluid Mech. (1975), *vol.* 71, *part* 3, *pp.* 465–479
Printed in Great Britain

Similarity behaviour of momentumless turbulent wakes

By MICHAEL L. FINSON
Avco Everett Research Laboratory, Inc., Everett, Massachusetts 02149†

(Received 14 August 1974)

Similarity solutions are determined for the turbulent wake of a self-propelled body (thrust = drag). The momentumless wake is shown to behave in a manner intermediate to homogeneous grid turbulence and more familiar free-shear flows such as the drag wake or jet. In essence the decay of momentumless-wake turbulence is similar to that of grid turbulence, but proceeds at a somewhat greater rate owing to lateral diffusion. The mean velocity difference is coupled to the difference $\overline{u^2} - \overline{v^2}$ between the axial and radial components of the mean-square fluctuating velocity. It is necessary to consider governing relations for various second-order turbulence quantities. Previously developed closure approximations yield far-wake decay rates that agree well with available measurements. Production of turbulent energy is negligible asymptotically; thus there is no balance between production and dissipation, and the far-wake behaviour does not become independent of the initial (near-wake) conditions. Even the radial profiles depend on the initial conditions, and there is no natural length scale with which to characterize the far wake.

1. Introduction

The evolution of turbulent wakes and jets has received much attention in connexion with many applications. A fundamental characteristic of such flows is the net momentum flux, which is (say) positive in a jet and negative in the wake of an unpropelled object. For a self-propelled body the thrust and drag are equal and there is no momentum excess or deficit in the wake. Such a wake represents a singular situation which has received little attention. As we shall see, the momentumless wake behaves rather differently from more familiar types of free-shear turbulence.

In many respects the momentumless turbulent wake considered here represents a rather idealized flow situation. Only the wakes of axisymmetric non-lifting bodies will be considered. While the extension to two-dimensional cases would be straightforward, the present analysis is not applicable to lifting bodies, whose wakes are dominated by trailing-edge vortices. Second, for the wake to be momentumless it is necessary not only that the body be self-propelled (thrust = drag), but also that all of the drag be contained within streamlines relatively near the body axis. For a self-propelled body at supersonic speeds (Newtonian pressure drag), or for a submarine or other submerged object within a few body

† Present address: Physical Sciences Inc., Wakefield, Massachusetts 01880.

diameters of the surface (Kelvin wave drag), a balance of momentum in the wake would not be achieved at any distance of interest. Further, we shall consider only the limit in which the Reynolds number tends to infinity. At any finite *Re*, direct viscous effects will enter beyond some downstream distance where the turbulence has decayed to such an extent that the eddy viscosity is not much greater than the molecular viscosity. Other effects not considered include ambient turbulence or stratification, buoyancy, swirl and compressibility.

2. The difficulty with the momentumless wake

For most types of free turbulent flow, derivation of the asymptotic similarity laws is straightforward (see, for example, Townsend 1956, p. 169). One separates the mean velocity into the free-stream velocity plus the velocity difference, and seeks a separable solution

$$U = U_0 + U_d(x) f(r/r_w), \tag{1}$$

where r_w is some measure of the width of the turbulent zone. With the boundary-layer approximation, neglect of mean pressure gradients ($P \cong P_0$) and the assumption that the velocity difference is small ($U_d/U_0 \ll 1$), the mean momentum equation in the axial direction reduces to

$$U_0 dU_d/dx \propto \epsilon U_d/r_w^2. \tag{2}$$

As suggested by Prandtl, the eddy viscosity ϵ should be approximately constant across the turbulent zone and proportional to the velocity difference and the width: $\epsilon \propto U_d r_w$. Thus

$$U_0 dU_d/dx \propto U_d^2/r_w. \tag{3}$$

Equation (3) involves two unknowns, $U_d(x)$ and $r_w(x)$. A second equation is provided by conservation of overall momentum, which for the axially symmetric drag wake gives

$$\int_0^\infty U(U_0 - U)\, 2\pi r\, dr = \tfrac{1}{2} U_0^2 C_D A, \tag{4a}$$

or with (1),

$$U_d r_w^2 \cong \text{constant}. \tag{4b}$$

Equations (3) and (4*b*) then yield the well-known axisymmetric wake solution $r_w \propto x^{\frac{1}{3}}$, $U_d \propto x^{-\frac{2}{3}}$. The corresponding laws for the two-dimensional wake and the two-dimensional or axially symmetric jet may be obtained in a similar manner.

The right side of (4*a*) is zero for the momentumless wake and (4*b*) does not hold. A more careful examination of the basic equations is required to derive the proper integral relation. In so doing we apply the boundary-layer approximation $\partial/\partial r \sim r_w^{-1}$, $\partial/\partial x \sim x^{-1}$, $r_w/x \ll 1$ and the far-wake approximation $U_d/U_0 \ll 1$, but make no assumptions regarding the magnitude of the mean pressure or fluctuating velocities (see Tennekes & Lumley 1972, p. 104 for a detailed discussion). Considering first the momentum equation in the radial direction, we obtain a result that applies to all thin turbulent shear flows (primes are omitted from the fluctuating quantities):

$$P + \rho\overline{v^2} = P_0. \tag{5}$$

For the mean momentum equation in the axial direction, (5) may be used to eliminate $\partial P/\partial x$, giving

$$U\frac{\partial U}{\partial x}+\frac{\partial}{\partial x}(\overline{u^2}-\overline{v^2})+\frac{1}{r}\frac{\partial}{\partial r}(r\overline{uv}) = 0 \qquad (6)$$

(viscous diffusion is neglected in comparison with turbulent diffusion). The first term here is of order $U_0 U_d/x$, while the second is of order u_m^2/x if $u_m(x)$ denotes the maximum r.m.s. fluctuating velocity $(\overline{u^2})^{\frac{1}{2}}$ at x. In a drag wake, where

$$u_m \sim U_d \ll U_0,$$

the second term in (6) would be small. However, this is not the case for the momentumless wake. Integrating (6) across the wake yields

$$\int_0^\infty U(U_0-U)\,2\pi r\,dr = \int_0^\infty (\overline{u^2}-\overline{v^2})\,2\pi r\,dr. \qquad (7)$$

The term on the right side of this equation was tacitly neglected in comparison with $\frac{1}{2}U_0^2 C_D A$ in writing down (4a), since one would expect $u' \ll U_0$ at large x. Previous studies have not used the proper integral momentum relation. Birkhoff & Zarantonello (1957, p. 307) and Tennekes & Lumley (1972, p. 124) set the right side of (7) to zero, and obtained $r_w \propto x^{\frac{1}{5}}$, $U_d \propto x^{-\frac{4}{5}}$.

At this point it becomes necessary to consider governing relations for second-order turbulence parameters. Equation (7) indicates that the mean velocity is directly related to mean-square fluctuating velocities in the momentumless wake. The coupling between mean and higher-order turbulence parameters may not be describable solely through the eddy viscosity, and the usual closure approximations may not hold. From (7), the r.m.s. fluctuating velocity should vary as $u_m \sim (U_0 U_d)^{\frac{1}{2}}$. In most free-shear flows $u_m \sim U_d$ and the eddy-viscosity expression used above ($\epsilon \propto U_d r_w$) may be considered to be a closure approximation which is equivalent to $u_m \sim U_d$. Thus it may be impossible to obtain an adequate description of the momentumless wake by invoking closure at first order.

The measurements performed by Naudascher (1965) suggest further differences between the momentumless wake and other free-shear flows. Whereas in most free-shear flows the turbulent kinetic energy results from a balance between the effects of production, dissipation, diffusion and convection, Naudascher found that the production term becomes negligible beyond about 10 diameters downstream in the momentumless wake. This finding is not surprising since the production rate is proportional to the mean shear, and the velocity difference can decay quite rapidly when an integral relation such as (4a) does not hold. The following sections will attempt to describe the behaviour of turbulence in momentumless wakes in more detail.

3. Formulation for second-order turbulence quantities

The conservation equation for a general component $\overline{u_i u_j}$ of the turbulent Reynolds-stress tensor can be derived from the Navier–Stokes equation (see, for example, Rotta 1951; Hinze 1959, p. 250). For steady flow, neglecting viscous diffusion terms, the result is

$$U_k \frac{\partial \overline{u_i u_j}}{\partial x_k} = -\overline{u_k u_j}\frac{\partial U_i}{\partial x_k} - \overline{u_k u_i}\frac{\partial U_j}{\partial x_k} + \frac{1}{\rho}\overline{p\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)} - \frac{\partial}{\partial x_k}(\overline{u_k u_i u_j}) - \frac{1}{\rho}\frac{\partial}{\partial x_k}\overline{(u_i\delta_{jk}+u_j\delta_{ik})p} - 2\nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}}. \quad (8)$$

The tensor summation convention is implied in (8). As generally interpreted, the first two terms on the right side describe production of $\overline{u_i u_j}$ by the mean shear. The third term on the right represents the action of pressure fluctuations to isotropize the turbulence; the fourth and fifth terms represent turbulent diffusion; and the final term describes viscous dissipation. In its present form, this equation is not useful since it contains additional unknown correlations of fluctuating quantities: in the pressure fluctuation, turbulent diffusion and dissipation terms. Closure approximations are required, and for these we follow other investigators of second-order closure such as Rotta (1951), Donaldson (1972) and Hanjalić & Launder (1972).

The description most commonly used for the dissipation term is

$$2\nu\overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}} = \tfrac{2}{3}k_d\frac{q^3}{\Lambda}\delta_{ij}, \quad (9)$$

where q^2 is the turbulent kinetic energy $\frac{1}{2}\overline{u_i u_i}$ and Λ the macroscale. This expression reflects the fact that dissipation should be isotropic at high Reynolds numbers. The constant k_d is presumed to be universal, although its value will not prove crucial in deriving similarity laws for the momentumless wake. Note that two more second-order correlations (q and Λ) are introduced by (9), and we shall require a conservation equation for Λ.

Pressure fluctuations drive the turbulence towards isotropy. Here we follow Rotta's (1951) suggestion, as have several others:

$$\frac{1}{\rho}\overline{p\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)} = -k_p\frac{q}{\Lambda}(\overline{u_i u_j} - \tfrac{2}{3}q^2\delta_{ij}). \quad (10)$$

An additional term involving the mean shear has been omitted for simplicity, since that term only supplements the production terms. The ratio k_p/k_d of the closure constants will prove to be important below.

The turbulent transport terms in (8) are of less concern here, since they represent neither sources nor sinks of the generalized Reynolds stress. For convenience we shall model them via gradient diffusion, with the diffusivity proportional to the eddy viscosity:

$$\overline{u_k u_i u_j} + \rho^{-1}\overline{p(u_i\delta_{jk}+u_j\delta_{ik})} = -C_\epsilon \epsilon\,\partial(\overline{u_i u_j})/\partial x_k. \quad (11)$$

Various authors (Donaldson 1972; Hanjalić & Launder 1972; Daly & Harlow 1970) have investigated more general tensor forms for gradient diffusion.

This completes the required closure approximations, except that a conservation equation is required for the scale size. Such an equation has been derived by Rotta (1951) and more recently by Finson (1973), who made use of Kolmogorov's universal-equilibrium hypothesis. The scale-size equation results from the governing equation for the spectrum of the kinetic energy; it is necessary to assume the spectrum to be isotropic (only one scale size Λ). With the further approximation that the spectra of the kinetic energy, production and dissipation have similar shapes in wavenumber space, the result is

$$q^2 U_k \frac{\partial \Lambda}{\partial x_k} = (1-d_2)\,\overline{u_i u_k}\,\frac{\partial U_i}{\partial x_k}\Lambda + (1-d_1)\,k_d q^3. \tag{12}$$

Here d_1 and d_2 are closure constants. Diffusion terms are neglected here, which is appropriate as the scale is observed to be approximately constant across most free-shear flows (e.g. Demetriades 1968).

Equations (6) and (8)–(12) represent the required governing relations, complete to second order in turbulence correlations. An appropriate set of dependent variables for the momentumless wake would be the mean velocity, the 'anisotropy' $\overline{u^2}-\overline{v^2}$, the kinetic energy q^2, the Reynolds stress $-\overline{uv}$ and the macroscale Λ. Of course, the equations for $\overline{u^2}-\overline{v^2}$, q^2 and $\overline{uv}$ can all be obtained from (8) by proper manipulation of the indices i and j.

4. Similarity solution

Axial dependence

We seek separable solutions, as already suggested by (1) for the mean velocity. For the other variables,

$$\left.\begin{aligned} \overline{u^2}-\overline{v^2} &= w_m^2(x)\,g(\eta),\\ q^2 = q_m^2(x)\,g(\eta), \quad \overline{uv} &= uv_m(x)\,h(\eta), \quad \Lambda = \Lambda(x), \end{aligned}\right\} \tag{13}$$

where $\eta = r/r_w$. Note that each component of $\overline{u_i u_i}$ is taken to have the same radial profile, consistent with use of the same diffusivity for each component in (11).

If (1) and (13) are inserted into the equation (6) for the mean velocity difference,

$$U_d'\left\{f - \frac{d\ln U_d}{d\ln r_w}\eta f'\right\} = -w_m^{2\prime}\left\{g - \frac{d\ln r_w}{d\ln w_m^2}\eta g'\right\} - \frac{uv_m}{r_w}\frac{1}{\eta}(\eta h)', \tag{14}$$

where a prime represents differentiation with respect to the argument. For a separable solution to exist, it is necessary that $d\ln r_w/d\ln U_d$, $d\ln r_w/d\ln w_m^2$, $U_d'/w_m^{2\prime}$ and $U_d'(uv_m/r_w)^{-1}$ all be constants. Note that the first two quantities would be constant for any power-law behaviour. Treating the other governing equations in a similar manner, additional ratios are found which must be constant for the existence of a separable solution. However, integration of the governing equations across the wake, when coupled with a relation for diffusion of mean momentum, provides an equivalent and more straightforward (but less formal) manner of determining the far-wake decay rates. The mean diffusion equation results from the condition of separability applied to (14):

$$dU_d/dx = \lambda uv_m/r_w, \tag{15a}$$

where λ is a separation constant. Integrating the conservation equations for U_d, q^2, $\overline{u^2}-\overline{v^2}$, $\overline{uv}$ and Λ across the wake yields

$$w_m^2 = \beta U_0 U_d, \tag{15b}$$

$$U_0\frac{d(q_m^2 r_w^2)}{dx} = -uv_m U_d r_w I_1 - k_d\frac{q_m^3}{\Lambda} r_w^2 I_2, \tag{15c}$$

$$U_0\frac{d(w_m^2 r_w^2)}{dx} = -uv_m U_d r_w I_1 - k_p\frac{q_m}{\Lambda} w_m^2 r_w^2 I_2, \tag{15d}$$

$$U_0\frac{d(uv_m r_w^2)}{dx} = -q_m^2 U_d r_w I_3 - k_p\frac{q_m}{\Lambda} uv_m r_w^2 I_4, \tag{15e}$$

$$U_0\frac{d\Lambda}{dx} = (1-d_2)\frac{uv_m U_d}{q_m^2 r_w}\Lambda I_1 + (1-d_1)k_d q_m I_2. \tag{15f}$$

Here we have six equations for six unknown functions of x. The first terms on the right sides of (15c–f) are production terms, while the second are sink terms (dissipation or pressure fluctuations). β and the I's represent radial profile factors:

$$\beta = -\int_0^\infty f\eta\, d\eta \bigg/ \int_0^\infty g\eta\, d\eta, \quad I_1 = \int_0^\infty hf'\eta\, d\eta \bigg/ \int_0^\infty g\eta\, d\eta, \quad \text{etc.}$$

Thus far we have retained the terms representing production of turbulent energy, although it has already been noted that such terms may be negligible in the well-developed momentumless wake. To estimate their importance we take the maximum kinetic energy q^2 and anisotropy $\overline{u^2}-\overline{v^2}$ at a distance x to be of order $u_m^2(x)$. From (7) the mean velocity difference U_d is of order u_m^2/U_0. We further take the macroscale Λ to be proportional to r_w, and assume that the eddy viscosity $\epsilon \sim u_m\Lambda$, so that $-\overline{uv} = \epsilon\partial U/\partial y \sim u_m^3/U_0$. With these values the dissipation terms (those containing I_2) in (15c,d,f) are of order $u_m^3 r_w$, while the production terms (involving I_1) are $\sim u_m^5 r_w/U_0^2$. Since $u_m^2/U_0^2 \to 0$ as $x\to\infty$, we may indeed neglect the production of q^2, $\overline{u^2}-\overline{v^2}$ and Λ in the asymptotic wake. However, the production and dissipation terms in (15e) are both of order $u_m^4 r_w/U_0$, so the production of Reynolds stress cannot be neglected. We shall proceed to determine the desired far-wake decay rates under the assumption that production is important only in the equation for $\overline{uv}$, to be verified *a posteriori*.

We now search for power-law solutions:

$$U_d \sim x^{n_1}, \quad q_m^2 \sim x^{n_2}, \quad w_m^2 \sim x^{n_3}, \quad uv_m \sim x^{n_4}, \quad r_w \sim x^{n_5}, \quad \Lambda \sim x^{n_6}. \tag{16}$$

Equating the axial dependence of the different terms (except those containing I_1) of each of (15a–f) results in four independent equations for the exponents. Integrating (15c,f) together yields a fifth relation, $n_2+2n_5 = -n_6/(1-d_1)$, and (15d,f) yield the sixth relation, $n_2+2n_5 = -(k_p/k_d)\,n_6(1-d_1)^{-1}$. Thus, in contrast

	Quantity	Momentumless wake	Drag wake
Velocity difference	n_1	$-1{\cdot}636$ $(-\frac{18}{11})$	$-0{\cdot}667$ $(-\frac{2}{3})$
Kinetic energy	n_2	$-1{\cdot}455$ $(-\frac{16}{11})$	$-1{\cdot}333$ $(-\frac{4}{3})$
Degree of anisotropy	n_3	$-1{\cdot}636$ $(-\frac{18}{11})$	$-1{\cdot}333$ $(-\frac{4}{3})$
Reynolds stress	n_4	$-2{\cdot}364$ $(-\frac{26}{11})$	$-1{\cdot}333$ $(-\frac{4}{3})$
Wake width	n_5	$0{\cdot}276$ $(\frac{3}{11})$	$0{\cdot}333$ $(\frac{1}{3})$
Macroscale	n_6	$0{\cdot}276$ $(\frac{3}{11})$	$0{\cdot}333$ $(\frac{1}{3})$

TABLE 1. Power-law solutions ($d_1 = 0{\cdot}7$, $k_p/k_d = 1{\cdot}2$)

to most turbulent shear flows, the closure constants affect the asymptotic power laws; the results are

$$n_1 = n_3 = -\frac{4(1-d_1)+2k_p/k_d}{4(1-d_1)+1}, \quad n_2 = -\frac{4(1-d_1)+2}{4(1-d_1)+1}, \tag{17a,b}$$

$$n_4 = -\frac{6(1-d_1)+2k_p/k_d+1}{4(1-d_1)+1}, \quad n_5 = n_6 = \frac{2(1-d_1)}{4(1-d_1)+1}. \tag{17c,d}$$

This solution seems quite complicated, but it can be put into better perspective by comparison with the decay of grid turbulence. For homogeneous (but not necessarily isotropic) grid turbulence, the governing relations are far more simple than (15):

$$U_0\frac{dq^2}{dx} = -k_d\frac{q^3}{\Lambda}, \quad U_0\frac{dw^2}{dx} = -k_p\frac{q}{\Lambda}w^2, \tag{18a,b}$$

$$U_0 d\Lambda/dx = (1-d_1)k_d q. \tag{18c}$$

These yield the following power-law solution:

$$q^2 \propto x^{-1/(\frac{3}{2}-d_1)}, \quad w^2 \propto x^{-k_p/k_d(\frac{3}{2}-d_1)}, \quad \Lambda \propto x^{(1-d_1)/(\frac{3}{2}-d_1)}. \tag{19a–c}$$

We may turn to grid-turbulence measurements to establish the closure constant d_1 and the ratio k_p/k_d. (Presumably, the second-order closure formulation has sufficient universality to be applicable to both homogeneous turbulence and shear turbulence.) Comte-Bellot & Corrsin (1966, 1971) have compiled a considerable quantity of grid-turbulence data. They found that their data and those of others gave $q^2 \propto x^{-(1{\cdot}25\pm0{\cdot}05)}$ and $\Lambda \propto x^{0{\cdot}35-0{\cdot}40}$. Thus we take $d_1 = 0{\cdot}70 \pm 0{\cdot}05$. Similarly, the Comte-Bellot & Corrsin data indicate $w^2 \propto x^{-1{\cdot}50}$, which, from (19c), suggests $k_p/k_d = 1{\cdot}2$. This ratio must be considered to be less certain than the value determined for d_1. Rotta (1951) and Hanjalić & Launder (1972) have determined larger values for k_p/k_d in the range 2·5–3·0. However, their results were based on data obtained in strongly contracted ducts, and it may be that the contractions cause local anisotropy in the dissipation rate which complicates the evaluation of the pressure fluctuation term.

Table 1 presents the numerical values for the power-law solutions. The fractional values indicated are obtained by expressing the closure constants d_1 and k_p/k_d as rational fractions (e.g. $d_1 = \frac{3}{10}$). Also shown for comparison in table 1 are the appropriate values for the drag wake. Note that the momentumless and drag wakes do not differ significantly regarding the behaviour of the kinetic

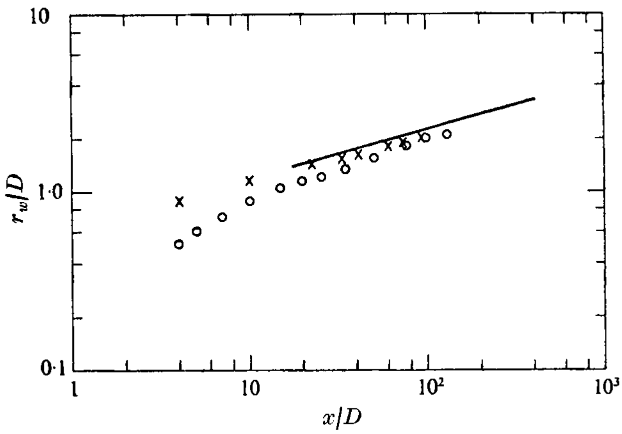

FIGURE 1. Comparison of similarity solution with the measurements of Naudascher (1965) and Merritt (1974) for the wake width. ——, $x^{\frac{3}{11}}$; ○, Naudascher; ×, Merritt.

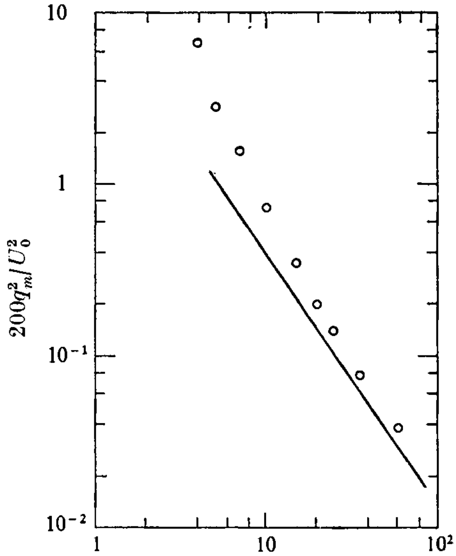

FIGURE 2

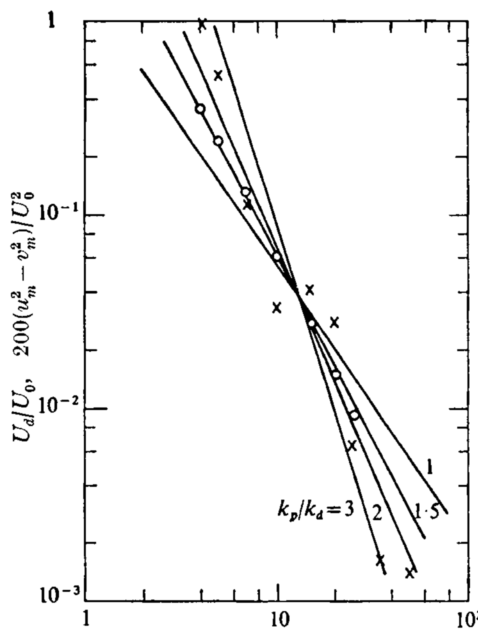

FIGURE 3

FIGURE 2. Comparison of similarity solution $x^{-\frac{16}{11}}$ with the measurements of Naudascher (1965) for the turbulent energy.

FIGURE 3. Comparison of similarity solution with the measurements of Naudascher (1965) for the mean velocity difference (circles) and the difference $u_m^2 - v_m^2$ between the axial and radial components of the mean-square fluctuating velocity (crosses).

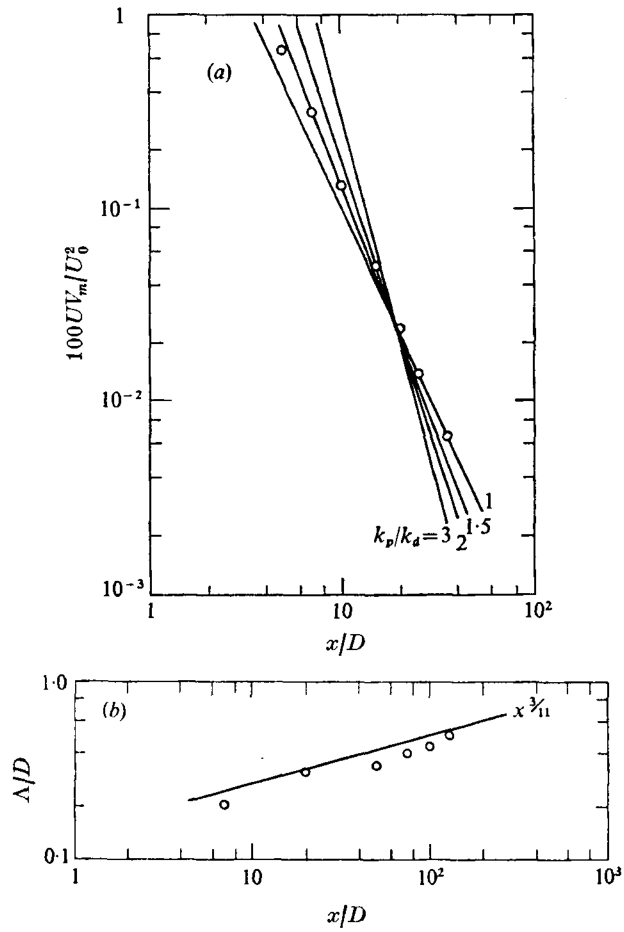

FIGURE 4. Comparison of similarity solution with the measurements of Naudascher (1965) for (a) the Reynolds stress and (b) the turbulent macroscale.

energy (n_2), wake width (n_5) or macroscale (n_6), but the velocity difference (n_1), degree of anisotropy (n_3) and Reynolds stress (n_4) decay much more rapidly in the momentumless wake.

Figures 1–4 compare the predicted far-wake behaviour with Naudascher's data. Evidently the growth or decay rates approach similarity values 20–40 diameters behind the body, although this distance could depend on the configuration of the generating body. Merritt's (1974) wake-width measurements, obtained visually from a wake into which dye was injected, are included in figure 1. It is somewhat unfortunate that experimental inaccuracies are greatest for the quantities U_d and w_m^2, which are the most sensitive tests of the theory. Owing to the differencing required, data cannot be obtained at large downstream distances for either quantity. We claim that the comparisons with the data for U_d and w_m^2 are adequate for the range of values of k_p/k_d indicated.

Finally, if the omitted production terms are evaluated using the solution obtained here, it is found that such terms decay at a greater rate than do the convection and dissipation terms as long as $k_p/k_d \gtrsim 0{\cdot}2$. It seems certain that $k_p/k_d \gtrsim 1{\cdot}0$, so the power-law solution expressed by (17) is a self-consistent solution to (15). It might also be noted that a power-law solution cannot be obtained from (15) if all the production terms are presumed to be important. Physically, this results from the fact that (7) requires the mean velocity to decay too rapidly for production of turbulent energy to remain important. Nor does a power-law solution result if *all* the production terms are dropped from (15): the decay of the Reynolds stress is sufficiently more rapid than that of the kinetic energy for the source term to remain in balance in (15*e*).

Radial profiles

We now turn to the other half of the separable solution: the determination of the self-preserving radial profiles for the various quantities. Equation (14) above is the governing relation for the mean velocity profile $f(\eta)$, and corresponding equations for the radial variation of q^2 or $\overline{u^2}-\overline{v^2} = w^2$ and $\overline{uv}$ can be obtained from (8)–(11). It is readily apparent from (14) that these equations will be coupled, and that complicates the determination of $f(\eta)$, $g(\eta)$ and $h(\eta)$. However, with the additional approximation that the eddy viscosity is constant across the wake, the equations can be decoupled to such an extent that $f(\eta)$ and $g(\eta)$ can be derived straightforwardly. We shall not attempt to determine the Reynolds-stress profile $h(\eta)$.

According to Naudascher's measurements the eddy-viscosity coefficient is roughly constant over the inner portion of the wake. Comparable observations have been made in most other types of free-shear turbulence. Thus we introduce

$$\overline{uv}(x,r) = \epsilon(x)\,\partial U_d/\partial r$$

into (14). With some straightforward rearrangement, the result may be expressed as

$$\frac{d^2f}{d\eta^2} + \left(\frac{1}{\eta} + \gamma\lambda\eta\right)\frac{df}{d\eta} - \lambda f = \beta\lambda\left(g - \lambda\eta\frac{dg}{d\eta}\right). \tag{20}$$

We have already introduced a gradient-diffusion model [equation (11)] for each component of the turbulent kinetic energy. Because we chose the diffusivity to be the same for each component, and also because the production terms are negligible asymptotically, the same radial profile $g(\eta)$ applies to any component of the kinetic energy, as already implied by (13). The governing relation for either q^2 or $w^2 = \overline{u^2}-\overline{v^2}$ leads to the following equation for $g(\eta)$:

$$\frac{d^2g}{d\eta^2} + \left(\frac{1}{\eta} + \frac{\gamma\lambda}{2C_\epsilon}\eta\right)\frac{dg}{d\eta} + 2\frac{\gamma\lambda}{2C_\epsilon}g = 0. \tag{21}$$

We recall that C_ϵ is the ratio of the diffusivity of kinetic energy to that of mean momentum, λ is an unknown separation constant [equation (15*a*)] and β is defined by conservation of mean momentum [equation (15*b*)]:

$$\beta = \frac{w_m^2}{U_0 U_d} = \int_0^\infty f\eta\,d\eta \Big/ \int_0^\infty g\eta\,d\eta. \tag{22}$$

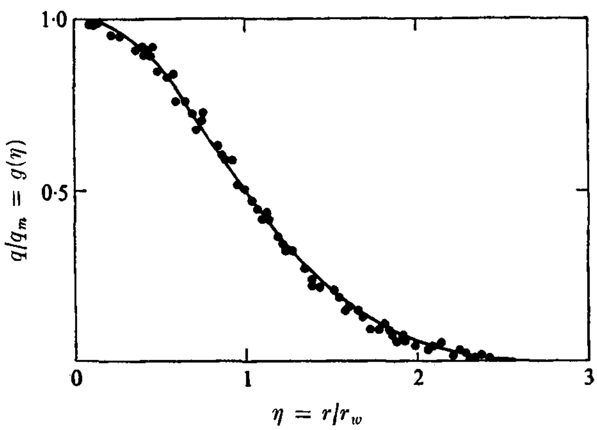

FIGURE 5. Radial profile for the kinetic energy: comparison with Naudascher's (1965) data. ——, predicted Gaussian profile.

Furthermore
$$\gamma = \frac{d\ln r_w}{d\ln U_d} = -\frac{1-d_1}{2(1-d_1)+k_p/k_d}. \tag{23}$$

Equation (21) for the kinetic-energy profile is homogeneous, and with the boundary conditions $g(0) = 1$ and $g \to 0$ as $\eta \to \infty$, the solution can easily be shown to be a Gaussian profile
$$g(\eta) = \exp\left(-\frac{\gamma\lambda}{4C_\epsilon}\eta^2\right). \tag{24}$$

So far the wake radius r_w ($\eta = r/r_w$) has not been precisely defined. If, as did Naudascher (1965), we define r_w as the radial location where the r.m.s. velocity or q is half its value on the axis, then $g(1) = 0{\cdot}25$ and we must have
$$\lambda = 8\ln 2C_\epsilon/\gamma. \tag{25}$$

Thus the separation constant is determined from requirements of self-consistency of definition. In figure 5 we compare the Gaussian profile with Naudascher's measurements at downstream distances in the range $5 \leqslant x/D \leqslant 50$. As Naudascher recognized, the kinetic-energy profile is very well represented by a Gaussian profile.

Equation (20) shows that $\overline{u^2}-\overline{v^2}$ serves as an inhomogeneous source term for the mean velocity profile. If we introduce $z = -\frac{1}{2}\gamma\lambda\eta^2$, (20) becomes
$$z\frac{d^2f}{dz^2}+(1-z)\frac{df}{dz}+\frac{1}{2\gamma}f = -\frac{\beta}{2\gamma}\left[g-2\gamma z\frac{dg}{dz}\right], \tag{26}$$

where $g(z) = \exp(z/2C_\epsilon)$. The homogeneous solution is a confluent hypergeometric function, often denoted by $M(-(2\gamma)^{-1}, 1, z)$ or $\Phi(-(2\gamma)^{-1}, 1, z)$. The complementary solution involves a logarithmic singularity at the origin (Abramowitz & Stegun 1964, p. 504), which is inconsistent with the boundary condition $f(0) = 1$. We have not succeeded in obtaining a complete solution to (26) in terms

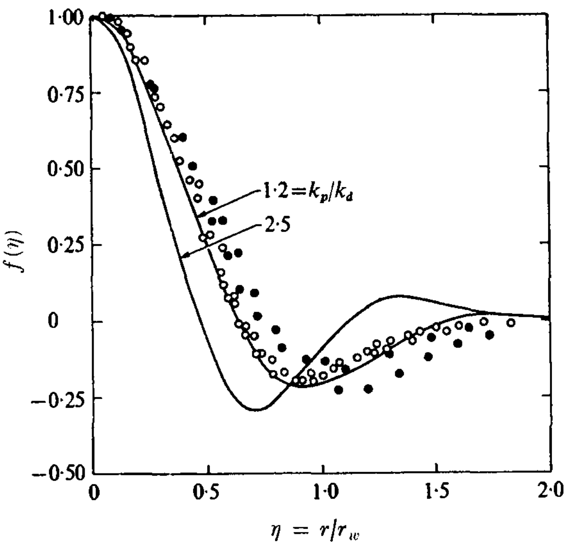

FIGURE 6. Radial profiles for the mean velocity difference. ——, present theory, $\beta = 0{\cdot}03$, $2C_\epsilon = 1$. Naudascher's data: ●, $x/D \leqslant 7$; ○, $x/D \geqslant 10$.

of analytical functions. However, a series solution can be determined in a straightforward manner:

$$\left.\begin{aligned} f(z) &= \sum_{k=0} A_k z^k, \\ A_{k+1} &= \frac{k-(2\gamma)^{-1}}{(k+1)^2}\left[A_k + \frac{\beta}{(2C_\epsilon)^k k!}\right], \\ A_0 &= 1. \end{aligned}\right\} \qquad (27)$$

This reduces to the series for the confluent hypergeometric function for $\beta = 0$.

In Naudascher's experiment, the appropriate value of β was approximately 0·03. For such a small value of β, the radial profiles differ negligibly from the homogeneous solution. Figure 6 compares Naudascher's data with the profiles computed numerically from (27) with $2C_\epsilon = 1$ and $k_p/k_d = 1{\cdot}2$ and 2·5. For $k_p/k_d = 1{\cdot}2$, the comparison is quite good; for $k_p/k_d = 2{\cdot}5$, rather good agreement would have resulted from a smaller value of $2C_\epsilon$ ($\simeq 0{\cdot}5$). The computed profiles exhibit a slight positive overshoot at large η, due perhaps to the crudity of the present treatment at the outer edges of the wake, where intermittency effects are large. It is also worth noting that, strictly speaking, the radial profile for the mean velocity depends upon β. This quantity is determined by the initial conditions, with the result that the radial velocity profile may not be exactly the same for all momentumless wakes.

5. Discussion

The momentumless wake represents a type of turbulent flow which is intermediate to homogeneous grid turbulence and the more familiar type of free-shear turbulence such as the drag wake, jet and mixing layer. In essence, the decay of turbulence in a momentumless wake is very similar to the decay of grid turbulence, but proceeds at a somewhat greater rate owing to lateral diffusion. To show this explicitly we examined the limiting case of a wake where there is no mean shear and where the turbulence is initially isotropic.† In terms of the above equations, this corresponds to the limit $\beta \to 0$. Of course equation (15*e*) for the Reynolds stress is degenerate and it is necessary to introduce an additional relation for the diffusivity of the turbulent kinetic energy. This may be done by assuming $\epsilon \propto q\Lambda$ (as have several authors) or by going to third-order closure (as modelled by Hanjalić & Launder 1972). By either approach, it can be shown that the exponents for the power-law decay of the kinetic energy (n_2) and the growth of the wake width (n_5) and macroscale (n_6) are precisely as given by (17*b*, *d*) above. Thus the decay of turbulent energy in and the lateral spreading of a momentumless wake are completely independent of mean shear effects.

It may further be shown that the momentumless wake does not possess the strong degree of similarity which characterizes the drag wake (or jet or mixing layer). As in other free-shear flows, there is an asymptotic separable solution for the momentumless wake. But this solution involves the initial (i.e. near-wake) conditions, in contrast to other types of flow. In the asymptotic drag wake there is a balance between turbulent production and dissipation. The effects of the initial conditions disappear, and there is a natural velocity scale U_0 and a natural length scale $(C_D A)^{\frac{1}{2}}$ with which to normalize all quantities. For example it is well known that the velocity difference and wake width obey the following relations in the asymptotic turbulent drag wake:

$$\frac{U_d}{U_0} = K_1\left(\frac{x}{(C_D A)^{\frac{1}{2}}}\right)^{-\frac{2}{3}}, \quad \frac{r_w}{(C_D A)^{\frac{1}{2}}} = K_2\left(\frac{x}{(C_D A)^{\frac{1}{2}}}\right)^{\frac{1}{3}}, \tag{28}$$

where K_1 and K_2 are universal constants.

Because the production terms are asymptotically negligible for the momentumless wake (except in the Reynolds-stress equation), initial values of the turbulence parameters affect the far-wake solution. This is apparent from the manner in which (15) had to be solved. Without the production terms, only four relations for the six unknown exponents $n_1, \ldots, n_6$ result from comparing the various terms in each of (15). The other two equations have to be obtained by integrating (15*c*) in conjunction with (15*d*);

$$w_m^2 r_w^2 / w_{m,i}^2 r_{w,i}^2 = (q_m^2 r_w^2 / q_{m,i}^2 r_{w,i}^2)^{k_p/k_d}, \tag{29a}$$

and (15*c*) in conjunction with (15*f*),

$$\Lambda/\Lambda_i = (q_m^2 r_w^2 / q_{m,i}^2 r_{w,i}^2)^{1/(1-d_1)}, \tag{29b}$$

where a subscript i indicates an initial (near-wake) value. These initial values propagate through the solution, appearing in the factors multiplying the powers of x. Mathematical complexity prevents us from explicitly deriving these factors.

The fact that there is no natural length scale in the momentumless wake analogous to $(C_D A)^{\frac{1}{2}}$ in the drag wake can be demonstrated formally. If an integral

† The author is indebted to Professor L. S. G. Kovasznay for suggesting this exercise.

moment of the mean velocity were conserved, then a length scale L could be defined by

$$U_0^2 L^{m+1} = U_0 U_d \int_0^\infty f(r/r_w)\, r^m\, dr = U_0 U_d r_w^{m+1} \int_0^\infty f(\eta)\, \eta^m\, d\eta. \tag{30}$$

The $m = 1$ moment yields $(C_D A)^{\frac{1}{2}}$ for the drag wake. Taking the r^m moment of the mean momentum equation (with the eddy-viscosity approximation for the turbulent transport term) yields

$$U_0 \frac{d}{dx}\left\{U_d r_w^{m+1} \int_0^\infty f\eta^m\, d\eta\right\} = \epsilon U_d r_w^{m-1}(m-1)^2 \int_0^\infty f\eta^{m-2} d\eta - \frac{d}{dx}\left\{w_m^2 r_w^{m+1} \int_0^\infty g\eta^m\, d\eta\right\}. \tag{31}$$

The only possible power which would yield a constant value for the integral in (30) is $m+1 = -n_1/n_5$. From the power-law solution presented above this corresponds to $m = -1-\gamma^{-1} = 1+(k_p/k_d)(1-d_1)^{-1}$, or $m \simeq 5$ for $k_p/k_d \simeq 1{\cdot}2$ and $1-d \simeq 0{\cdot}3$. With this value of m, the products $U_d r_w^{m+1}$ and $w_m^2 r_w^{m+1}$ are independent of x. Since $g(\eta)$ is Gaussian, the integral in the last term in (31) has a constant, finite value and the derivative yields zero for that term. However the other term on the right side is not zero for $m \neq 1$. This proves that there is no unique length L as defined by (30), but it also introduces a paradox since m has been chosen such that $U_d r_w^{m+1}$ = constant. The paradox is resolved by noting that the integral $f\eta^m d\eta$ does not converge for $m = -1-\gamma^{-1}$. For large η the homogeneous portion of the solution for $f(\eta)$, the confluent hypergeometric function, goes as $\eta^{1/\gamma}$ (Abramowitz & Stegun 1964, p. 508). Thus the integrand $f(\eta)\,\eta^m \sim \eta^{-1}$ and the integral is not finite.

Finally, we might comment on the fact that a rather involved set of second-order turbulence parameters had to be considered in order to derive the basic similarity laws for the momentumless wake. It was claimed that the set chosen above, $\overline{u^2}-\overline{v^2}$, q, $\overline{uv}$ and Λ, was sufficient, and it is possible to examine the solution in retrospect to determine whether this set is necessary or whether some approximations might have been introduced for some of the parameters. Of course, the form of the mean momentum equation dictates computing $\overline{u^2}-\overline{v^2}$, and it is hard to imagine going to second order without solving for the kinetic energy q^2. On the other hand, our solution indicates that $n_5 = n_6$, or Λ/r_w = constant. This might have been anticipated from experience with other free-shear flows, and suggests that one might have dispensed with a conservation equation for Λ in favour of $\Lambda = Kr_w$. However if one attempts to derive the similarity laws along such lines, it is found that the constant K appears in the power-law exponents at the expense of the quantity $1-d_1$. Whereas $1-d_1$ can be fixed with some precision by analysis of grid-turbulence data, the value of the constant K is not accurately known for drag wakes and could not be extrapolated to the momentumless wake with any confidence. In fact, Naudascher (1965) attempted to derive similarity laws using second-order modelling but failed to obtain concrete results because he did not consider the scale-size equation. The one equation in our set which is perhaps superfluous is that for the Reynolds stress $-\overline{uv}$. The solution indicates that $\epsilon \propto q\Lambda$ and one could have introduced the approximation $\epsilon \propto q\Lambda$ without the constant of proportionality affecting the solution in a fundamental manner. It is interesting to note that this proportionality is assured by the presence of the production term in the Reynolds-stress equation.

REFERENCES

ABRAMOWITZ, M. & STEGUN, I. A. 1964 *Handbook of Mathematical Functions*. Washington: Nat. Bur. Stand.

BIRKHOFF, G. & ZARANTONELLO, E. H. 1957 *Jets, Wakes, and Cavities*. Academic.

COMTE-BELLOT, G. & CORRSIN, S. 1966 *J. Fluid Mech.* **25**, 657.

COMTE-BELLOT, G. & CORRSIN, S. 1971 *J. Fluid Mech.* **48**, 273.

DALY, B. J. & HARLOW, F. H. 1970 *Phys. Fluids*, **13**, 2634.

DEMETRIADES, A. 1968 *Phys. Fluids*, **11**, 1841.

DONALDSON, C. DUP. 1972 *A.I.A.A. J.* **10**, 4.

FINSON, M. L. 1973 *A.I.A.A. J.* **11**, 1137.

HANJALIĆ, K. & LAUNDER, B. E. 1972 *J. Fluid Mech.* **52**, 609.

HINZE, J. O. 1959 *Turbulence*. McGraw-Hill.

MERRITT, G. E. 1974 *A.I.A.A. J.* **12**, 940.

NAUDASCHER, E. 1965 *J. Fluid Mech.* **22**, 625.

ROTTA, J. 1951 *Z. Phys.* **129**, 547.

TENNEKES, H. & LUMLEY, J. L. 1972 *A First Course in Turbulence*. M.I.T. Press.

TOWNSEND, A. A. 1956 *The Structure of Turbulent Shear Flow*. Cambridge University Press.